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# Quantum Insights from Null-Strut Geometrodynamics

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We discuss quantum insights due to the null-strut formalism. These insights deal primarily with two topics: the formalism of a theory of canonical simplicial quantum gravity based on the geometrodynamic duality of null-strut calculus, and the natural implementation of spinors and spin networks in null-strut calculus.

## 1. INTRODUCTION

Quantum gravity has remained an elusive dream of physicists for several decades, always producing more questions than answers. The perturbative Feynman diagram method, which had been so successful in dealing with quantum electrodynamics, was not sufficient to deal with gravity. Diagrams with loops in the expansion of quantum gravity have yielded divergences which cannot be renormalized with a finite number of parameters. However, quantum gravity is, by nature, a nonperturbative theory. All terms in the loop expansion are equally important. Many physicists now believe that the renormalization problem may be due to the inappropriate use of perturbative methods. This is the major motivation behind the development of nonperturbative methodologies for quantum gravity calculations.

A nonperturbative method which has recently been gaining in popularity is the simplicial minisuperspace method. A description of the method, in the Lagrangian formalism, has been given (Hartle, 1985) and a computer

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calculation for gravity with higher derivative terms has been accomplished (Hamber and Williams, 1984, 1985, 1986a,b). The formalism given by Hartle is described briefly as follows.

Consider the wave function $\Psi_0$ for a closed three-geometry in a state of minimum excitation. The three-geometry consists of $n$ disconnected compact three-manifolds $\partial M_i$, which are without boundary, having three-metrics $h_i$. These manifolds might also have nontrivial topology. $\Psi_0$ is given by

$$\Psi_0[h_i, \partial M_i; i=1, n] = \sum_M \nu(M) \int_C \delta g_{\alpha\beta} \exp\{-I[g_{\alpha\beta}, M]\} \tag{1}$$

where the sum is over four-manifolds $M$, with the boundary manifold stated above, and each one contributing with weight $\nu(M)$; $g_{\alpha\beta}$ is a Euclidean four-geometry on the manifold $\partial M$; and $I$ is the Euclidean gravitational action, including cosmological constant and surface terms

$$l^2 I[g_{\alpha\beta}] = -2 \int_{\partial M} d^3x\, h^{1/2} K - \int_M d^4x\, g^{1/2}(R - 2\Lambda) \tag{2}$$

We are using units where $h = c = 1$. Therefore, the Planck length is given by $l = (16\pi G)^{1/2}$ (see Figure 1).

Let us now approximate $M$ with metric $g_{\alpha\beta}$ by a triangulation with vertices $\Sigma_0$, edges $\Sigma_1$, triangles $\Sigma_2$, tetrahedra $\Sigma_3$, and four-simplices $\Sigma_4$ ($\Sigma_\alpha$). The boundary simplices, which approximate $(h_i, \partial M_i)$, are denoted by $\partial\Sigma_{\alpha i}$ and the interior simplices are denoted by $\text{int}\,\Sigma_\alpha$. Let $s_j$ denote the squared edge lengths of the lattice which are projections of the metric. In

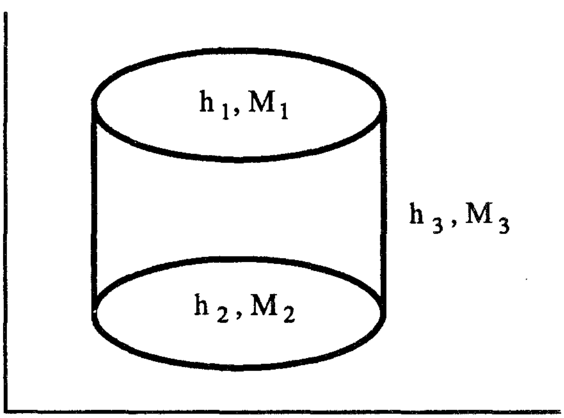


**Fig. 1.** A representation of the configuration space for path integral quantization.

the simplicial approximation, (1) is replaced by

$$\Psi_0[s_j, j \in \partial\Sigma_{\alpha i}] = \sum_M \nu(M) \int_C d\Sigma_1 \exp[-I(s_j)] \tag{3}$$

with the action replaced by the Regge action (with surface terms)

$$I = -2 \sum_{\sigma \in \partial\Sigma_2} A(\sigma)\psi(\sigma) - 2 \sum_{\sigma \in \text{int}\,\Sigma_2} A(\sigma)\theta(\sigma) + 2\Lambda \sum_{\sigma \in \Sigma_4} V_4(\tau) \tag{4}$$

where $A(\sigma)$ is the area of triangle $\sigma$, and $V_4(\tau)$ is the volume of the four-simplex $\tau$. The angle $\theta(\sigma)$ is the deficit angle for triangle $\sigma$, given by

$$\theta(\sigma) = 2\pi - \sum_\tau \theta(\sigma, \tau) \tag{5}$$

where the sum is over all four-simplices that contain $\sigma$, and $\theta(\sigma, \tau)$ is the dihedral angle between the two tetrahedra of $\tau$ that have $\sigma$ as a common face. The angle $\psi(s)$ is given by

$$\psi(\sigma) = \pi - \sum_\tau \theta(\sigma, \tau) \tag{6}$$

where the sum is over all four-simplices that intersect a triangle on the boundary $\sigma$.

We may use (3) to calculate interesting expectation values of $\Psi_0$:

$$\langle A \rangle = \frac{\int d(\partial\Sigma_1)\Psi_0(s_j)A(s_j)\Psi_0(s_j)}{\int d(\partial\Sigma_1)\Psi_0(s_j)\Psi_0(s_j)} \tag{7}$$

where $d(\partial\Sigma_1)$ is the volume element on the space of boundary edge lengths. If $d(\partial\Sigma_1)$ is consistent with $\mathrm{d}\Sigma_1$, then (7) becomes

$$\langle A \rangle = \frac{\int_c d\Sigma_1\, A(s_j) \exp[-I(s_j)]}{\int_c d\Sigma_1 \exp[-I(s_j)]} \tag{8}$$

where the integral is now over the space of squared edge lengths of the compact, boundaryless, manifold formed from $M$ and a copy of itself attached at the boundary.

The lattice link lengths of the theory act as ultraviolet cutoffs, thus providing a regularization mechanism. Simplicial quantum gravity has the additional advantage of being able to handle a very general set of metrics (squared link lengths) having no built-in symmetry constraints. In analytic methods, the metrics are limited to some class of metrics described by a small set of parameters (e.g., Robertson–Walker metrics or Bianchi I metrics), thus ignoring a wide variety of contributions to the path integral. Furthermore, since the theory is so general, manifolds with different topologies (and even nonmanifold configurations) can be represented easily in their simplicial approximations.

Although some work has been done, such as the Monte Carlo calculation of Hamber and Williams, questions remain to be dealt with before this program can be expected to give reasonable results. Some of the questions are of a practical nature, such as how to represent a large enough Regge lattice on the computer to approximate a reasonable spacetime. However, there are other questions that are of a fundamental nature:

1. Can the Wick rotation of the integration contour from Lorentzian to Euclidean signature be justified for gravity?
2. How can the indefiniteness of the gravitational action (and also the Regge action) be dealt with in a physically reasonable way?
3. What is the form of the integration measure for simplicial quantum gravity?
4. What are the boundary conditions for the space of dynamic variables in quantum gravity?
5. What, if any, is the lattice version of the diffeomorphism group? How can gauge fixing and ghosts, if actually necessary, be implemented in the simplicial theory?
6. Is the spacetime at the quantum level better described by a discrete structure (e.g., Regge lattice) than by the usual continuum manifold? In other words, is simplicial quantum gravity an actual theory of the small structure of spacetime or is it just an approximation of the continuum quantum theory?
7. How can a spinor structure be associated with a Regge lattice and its dynamic structure be developed? Can simplicial quantum gravity be rewritten in terms of spinor dynamical variables?

In the discussion below, we deal with insights into the formulation of simplicial quantum gravity in $3+1$ dimensions given by null-strut geometrodynamics. The insights will deal mostly with the basic structure of such a formulation, and possible solutions of the last two questions listed above. We shall not present the complete formalism of null-strut calculus in this paper, but refer the reader to the literature (Miller, 1985, 1986a,b). In Section 2, we discuss a basic structure for $(3+1)$-dimensional simplicial quantum gravity using the light cone induced duality of null-strut geometrodynamics. In Section 3, we outline the motivation for using spinors in general relativity and null-strut geometrodynamics. We discuss a natural formulation of a spinor structure on the null-strut lattice, using the Penrose interpretation of a spinor (Penrose and Rindler, 1984). Finally, in Section 4, we speculate on the possible result of quantizing the spinor structure of Section 3. We will outline a $(3+1)$-dimensional generalization of a theory of quantum gravity by Ponzano and Regge (1968). In this formulation, three-dimensional simplicial quantum gravity can be shown to be a special case of a spin network. The theory has some very appealing characteristics inherent in its structure which may make it useful in the quantum gravity program.

## 2. THE CANONICAL FORMULATION OF SIMPLICIAL QUANTUM GRAVITY

Given the covariant formalism outlined above, why would we want to develop a canonical version? First of all, the canonical formulation is closer to the nature of the Dirac version of quantum mechanics, providing both a motivation and framework for the development. The interpretation of initial conditions, matrix elements, and expectation values is easier in the canonical formalism because of the existence of a time parameter. Secondly, Kuchar (1986) showed that a canonical formalism could be implemented with holonomic constraints satisfying the Lie algebra of the diffeomorphism group. The recovery of the diffeomorphism group is a most compelling reason to believe that the canonical formalism and the covariant formalism are equally sound. Two of the dynamic variables are the usual 3-metric $\gamma_{ij}$ and the extrinsic curvature $k_{ij}$ of a hypersurface. Given the spatial coordinates $x^a$ on a hypersurface $\Sigma$ and the spacetime coordinates $X^\alpha$ on the spacetime manifold $M$, we can parametrize an embedding of $\Sigma$ in $M$ by the four functions $X^\alpha(x^a)$ such that the relationship between the two metrics is given by

$$\gamma_{ab}(x; X] = g_{\alpha\beta}(X(x))X^\alpha_{,a}(x)X^\beta_{,b}(x) \tag{9}$$

where the square bracket in $\gamma_{ab}(x; X]$ emphasizes that $\gamma_{ab}$ is a function of $x$ and a functional of $X$. The $X^\alpha$ and the momentum variables conjugate to them $P^\alpha$, along with a Lagrange multiplier $N^\alpha$ for the new embedding momentum constraint, form the remaining dynamic variables. The constraints of the resulting theory, as stated above, form a representation of the Lie algebra of the diffeomorphism group. The usual canonical formulation may be recovered by a suitable transformation.

One might wonder why we choose to use the null-strut formulation to model our dynamical variables, rather than a Regge lattice version of the method developed by Arnowitt *et al.*, 1962. The fully geometric nature of null-strut calculus, coupled to the natural geometric duality of the structure, make it very attractive for the interpretation of the dynamic constraints. Furthermore, using null-strut lattices as our basic 3+1 structure, we shall see that spinors, in the Penrose interpretation, may be elegantly integrated into the structure. The basic formalism of canonical simplicial quantum gravity in terms of null-strut calculus is given below.

Consider an initial ($t=0$) compact null-strut sandwich (TET-TET*)$_{\text{initial}}$ with a constant trace of the extrinsic curvature $\text{Tr}\, K = \text{const}$, and a final ($t=t'$) compact null-strut sandwich (TET-TET*)$_{\text{final}}$ with $\text{Tr}\, K = \text{const}'$ (initial and final spacelike hypersurfaces). We shall use the short-hand notation for these two slices given by (TT*)$_{\text{i}}$ and (TT*)$_{\text{f}}$, respectively. Connect these two slices with a series of compact null-strut sandwiches (TET$^*_p$-TET-TET$^*_f$)$_t$, if possible, such that the resulting spacetime is not necessarily a solution of the 3+1 Regge equations (TET equations and null-strut equations). We will label vertices in TET with $i, j, k, \ldots$, the edges in TET with $a, b, c, \ldots$, and tetrahedra in TET with $\alpha, \beta, \gamma, \ldots$. No labels for the triangles will be necessary.

This lattice four-geometry becomes a dynamic configuration in the canonical path integral, with dynamic variables given by the squared link lengths of TET($t$), $s_{at}$; the conjugate momenta (related to the extrinsic curvature $k_{at}$, which we will define below), given by $p_{at}$; the lapse function $N_{\alpha t}$, describing the local separation between two tetrahedra $\alpha$ on TET($t$) and the one on the next slice TET($t+\delta t$) along the timelike normal vector; and the shift vector $N^a_{\alpha t}$ describing the spacelike separation of these same two tetrahedra. The Euclidean canonical path integral for this system is given by

$$\Psi_0 = \prod_{a,\alpha,t} \int ds_{at}\, dp_{at}\, dN_{\alpha t}\, dN^a_{\alpha t} \exp\{-H[s_{at}, p_{at}, N_{\alpha t}, N^a_{\alpha t}]\} \tag{10}$$

where $H$ is the Hamiltonian for null-strut calculus. The Hamiltonian constraint in Regge calculus was given by Friedman and Jack (1986) as

$$-\frac{1}{4} \sum_{a,b\in\alpha} G^{(\alpha)}_{abt} p^a_t p^b_t - 2 \sum_{a\in\alpha} s^{1/2}_{at} \theta_{(\alpha a t)} = 0 \tag{11}$$

where

$$G^{(\alpha)}_{abt} = \frac{1}{V_{(\alpha t)} c_{(at)} c_{bt)}} G^{(\alpha)}_{abt} \tag{12}$$

and

$$\theta_{(\alpha a t)} = \frac{2\pi}{c(at)} - \delta_{(\alpha a t)} \tag{13}$$

$\delta_{(\alpha a t)}$ is the dihedral angle of tetrahedron $\alpha(t)$ in the entourage of edge $a(t)$, $c_{(at)}$ is the number of tetrahedra in the entourage of edge $a(t)$, $s^{1/2}_{at}$ is the link length of $a$, $V_{(\alpha t)}$ is the volume of the tetrahedron $\alpha(t)$, and $G^{(\alpha)}_{abt}$ is the inverse of the Regge calculus version of the DeWitt super metric

$$G^{(\alpha)ab}_t = -\frac{1}{V^2_{(\alpha t)}} \frac{\partial^2 V^2_{(\alpha t)}}{\partial s_{at}\, \partial s_{bt}} \tag{14}$$

The momentum constraint was given by

$$-2\Delta_n^\beta p^{mn} = 0 \tag{15}$$

where $\Delta_n^\beta$ is a finite-difference version of the gradient on a tetrahedral cell $\beta$. For any piecewise constant tensor, $\Delta_n^\beta$ is given by the expression

$$\Delta_n^\beta L_m = \frac{1}{2V_\beta} \sum_{\gamma \in {}^*\beta} A_n^{\beta\gamma} L_{\gamma m} \tag{16}$$

where $V_\beta$ is the volume of tetrahedral cell $\beta$, ${}^*\beta$ is the star of $\beta$ (the union of $\beta$ with the four tetrahedral cells adjoining it), and $A_n^{\beta\gamma}$ is the outward normal to the face $\beta \cap \gamma$ with magnitude equal to the area of the face.

From the constraints (11) and (15), we can form the Hamiltonian

$$H = \int dt \left[ \sum_a p_t^a \dot{s}_{at} - \sum_\alpha N_{\alpha t} \left( \frac{1}{4} \sum_{a,b \in \alpha} G_{abt}^{(\alpha)} p_t^a p_t^b + 2 \sum_{a \in \alpha} s_{at}^{1/2} \theta_{(\alpha a t)} \right) + \sum_{\alpha,a} \frac{1}{c_{(at)}} \Delta^\alpha N_{a_{\alpha t}} p_t^a \right] \tag{17}$$

where $\Delta^\alpha N_{a_{\alpha t}}$ is the projection of $\Delta_n^\alpha N_{m\alpha t}$ along the edge $l_a$ of $\alpha$, denoted by

$$\Delta^\alpha N_{a\alpha t} = \Delta_n^\alpha N_{m\alpha t} l_a^m l_a^n \tag{18}$$

and $\theta_{(\alpha a t)}$ is given by

$$\theta_{\alpha a t} = \frac{2\pi}{c_{(at)}} - \delta_{(\alpha a)} \tag{19}$$

where $\delta_{(\alpha a)}$ is the dihedral angle of the tetrahedron $\alpha$ on the edge $a$.

The following definition was used above:

$$\int_{V^{(\alpha t)}} N^{(3)} R g^{1/2} \, d^3x = 2 \sum_{a \in \alpha} N_{at} s_{at}^{1/2} \theta_{(at)} = 2N_{\alpha t} \sum_{a \in \alpha} s_{at}^{1/2} \theta_{(\alpha a t)} \tag{20}$$

where $N_{at}$ is defined to be

$$N_{at} \theta_{(at)} = \sum_{a \in \alpha} N_{\alpha t} \theta_{(\alpha a t)} \tag{21}$$

and we used the chain rule to obtain the last term in (17).

In order to evaluate the path integral (10) for our null-strut lattice, it is imperative that we know how the conjugate momentum $p_t^a$ manifests itself on the lattice. The $p_t^a$ is directly related to the extrinsic curvature projected along the link $a$, $k_t^a$, through the expression

$$p_t^a = -V_{(\alpha t)} (k_t^a - s_t^a \operatorname{Tr} k) \tag{22}$$

Therefore, if we know how the extrinsic curvature is defined on the null-strut lattice, we can use (22) to find the conjugate momentum. Knowledge of the extrinsic curvature on the lattice is also very important for defining the initial data for classical solutions of the Einstein equations.

We present a very simple, geometric method for extracting the extrinsic curvature associated with the triangles in TET. Consider the standard coordinate-free definition of extrinsic curvature given by

$$\mathbf{dn} = -\mathbf{K}(\mathbf{d}P) \tag{23}$$

where $\mathbf{n}$ is the unit normal vector at a point $P$ in a spacelike hypersurface, and $\mathbf{dn}$ is the difference between a unit normal vector at the point $P+\mathbf{d}P$ on the spacelike hypersurface and the unit vector obtained from the parallel transport of the unit normal vector at $P$ to $P+\mathbf{d}P$ [see Figure 2(a)].

Now consider the combination of null-strut building blocks in Figure 2(b). This combination, which we call a null-strut Marionette, is built from two wigwams $P_1$ and $P_2$ whose tetrahedral bases in TET share a common triangle $ABC$ and whose summits are connected by a link in TET* of length $L$. If we let $\mathbf{d}P$ be a vector from one summit to the other [see Figure 2(b)] and we are given an appropriate definition of parallel transport, we can use (22) to define the extrinsic curvature in terms of the angle $\psi$ between the wigwam altitude at $P+\mathbf{d}P$ and the other wigwam altitude parallel-transported from $P$ to $P+\mathbf{d}P$. The expression obtained is

$$K(L\hat{\Delta}_{ABC}) = -\frac{2}{\text{an appropriate length}} \sinh\frac{\psi}{2}\hat{\Delta}_{ABC} \tag{24}$$

where

$$\hat{\Delta}_{ABC} = \frac{\mathbf{AB}\wedge\mathbf{AC}}{2\Delta_{ABC}} \tag{25}$$

$\hat{\Delta}_{ABC}$ is a bivector dual to the unit vector $\hat{L}$ which points along $L$ from $P$, $\mathbf{AB}$ is a vector that points from $A$ to $B$, $\mathbf{AC}$ is a vector that points from $A$ to $C$, and $\Delta_{ABC}$ is the area of the triangle $ABC$. The "appropriate length" might be taken to be the distance between the barycenters of the two tetrahedra of the Marionette. This length is always well-defined for any Marionette. Work is continuing to determine if this is an adequate length for the denominator of (24) and this work will be published in an upcoming paper (Kheyfets *et al.*, 1988). In the affine basis of the null-strut construction, the extrinsic curvature defined in (24) is diagonal (as is the Einstein tensor). This is a great simplification and could provide needed shortcuts in performing computer calculations.

Now $\psi$ must be defined in terms of the geometry of the Marionette. To accomplish this, we need a well-defined parallel transport procedure. A

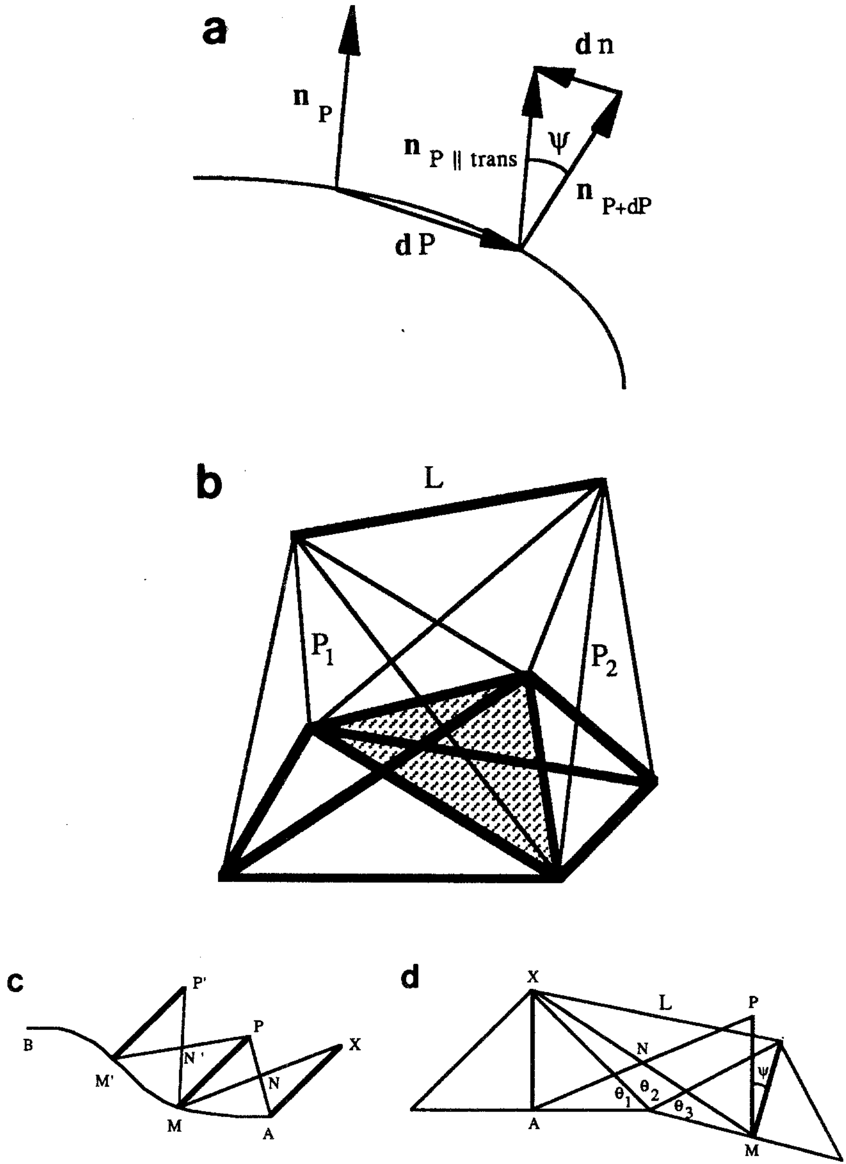


**Fig. 2.** (a) A graphical representation of the definition of extrinsic curvature in terms of the parallel transport of the normal from $P$ to $P+\mathbf{d}P$. (b) The carrier of information about the extrinsic curvature: the null-strut Marionette, composed of two wigwam blocks of a null-strut sandwich sharing a triangle in TET, and *a* the filler block between them. The extrinsic curvature is a function of the link lengths of the Marionette. (c) Parallel transport of a vector via the Schild's ladder. (d) Parallel transport of a wigwam altitude, in one-plus-one dimensional null-strut calculus, via the Schild's ladder.

simple geometric procedure is provided by the Schild's ladder [see Figure 2(c)]. Consider the parallel transport of the vector $AX$ (tangent vector) along the curve from $A$ to $B$ (Schild, 1970; Misner *et al.*, 1971). We may accomplish this transport in a series of small steps as follows. Choose a point $M$ close to $A$ along the curve from $A$ to $B$. Draw the geodesic $MX$ through $X$ and $M$. Take an affine parametrization $\lambda$ of $XM$ and use it to define a point $N$ by the condition

$$\lambda_N = \tfrac{1}{2}(\lambda_X + \lambda_M) \tag{26}$$

Draw the geodesic $AN$ from $A$ to $N$ and define a parameter $\rho$ along it. We may extend the geodesic $AN$ past $N$ an equal parameter increment to a point $P$. The vector $MP$ defines the parallel transport of $AX$ to the point $M$. We may repeat this procedure until the vector is transported, in small steps, to $B$. This construction also allows the definition of covariant derivatives in an unambiguous manner.

Figure 2(d) shows a Schild's ladder construction on a $(1+1)$-dimensional version of a null-strut marionette. We can see that, by geometry, the construction defines the angle $\psi$ in terms of the angles $\theta_1$, $\theta_2$, and $\theta_3$ as $\psi = \pi - (\theta_1 + \theta_2 + \theta_3)$. The same construction works for the $(3+1)$-dimensional marionette, where $\psi$ is given in terms of the hyperdihedral angles of the three building blocks of the marionette $\theta_{P_1}$, $\theta_{\text{filler wedge}}$, and $\theta_{P_2}$ as

$$\psi = \pi - (\theta_{P_1} + \theta_{\text{filler wedge}} + \theta_{P_2}) \tag{27}$$

Since $\theta_{\text{filler wedge}}$ is a function of $L$ and the TET links, we now know the dependence of the extrinsic curvature on the geometry of the Marionette through (23) and (27).

The extrinsic curvature may be defined in an alternative fashion, so that it is associated with the links of TET instead of the triangles of TET. This definition has two advantages. Terms in the Lagrangian density containing the extrinsic curvature and the metric may be defined unambiguously since they are associated with the same simplex (link). Second, the appropriate length in (24) becomes unambiguous, since the natural length scale is the link length.

Consider a vertex $X$ in the TET lattice. We may define a Marionette complex around $X$ by considering the union of all Marionettes that have $X$ contained in their tetrahedral bases. Since the interiors of the tetrahedral bases in TET are flat, the normals to all points in a given tetrahedron are parallel, forming a "brush bristle" structure. We may define a normal at $X$, $N(X)$, by the weighted average

$$N(X) = \sum_{i=1}^{N} V_i n_i / \text{normalization} \tag{28}$$

where $n_i$ is a normal on the $i$th tetrahedron of TET sharing $X$. This normal $n_i$ is defined as a unit vector along the wigwam altitude of this tetrahedral base (all other normals on the tetrahedron are parallel to this normal). $V_i$ is the volume of the $i$th tetrahedron. Define these normals $N(X)$ at every vertex $X$ of TET. The extrinsic curvature on a link may now be defined by parallel transport of $N(X)$ along the link 1 to the next vertex $(X+1)$ and using

$$K(l) = -\frac{2}{|l|}\sinh\frac{\psi}{2}\,\widehat{\mathbf{d}n} \tag{29}$$

where $l$ is the link, $|l|$ is the length of $l$, $\psi$ is the angle between the normal at $X+1$ and the parallel-transported normal from $X$ (via Schild's ladder), and $\widehat{\mathbf{d}n}$ is the unit vector along the difference vector $\mathbf{d}n$ between the normal at $X+1$ and the parallel transported normal from $X$. Note that the extrinsic curvature defined this way is no longer diagonal. Application of (29) to the case of a cylinder gives the correct expression for the extrinsic curvature in the continuum limit. Conceptually, this definition is dual to the first definition, and might be used to define the altitude of the fluted cone by restricting $N(X)$ to be the normalized vector from $X$ to the barycenter of the truncated octahedron.

The path integral (10), in principle, can be evaluated by a Metropolis algorithm, varying the dynamic variables $s_{at}$, $p_{at}$, $N_{\alpha t}$, and $N^a_{\alpha t}$ and calculating the corresponding Hamiltonian. The use of null-strut lattices has the appealing property of giving us back light cones (fluted cones and wigwams) which are lost in going from the Lorentzian to the Euclidean regime. The Hamiltonian is not positive-definite, and a methodology for handling this problem must be implemented to get convergence from the evaluation of (10). The addition of an $R$-squared term (split into $3+1$) would work (as was done by Hamber and Williams), or we could split the spacetime into conformal equivalence classes and distort the integration contour, as suggested by Hartle. Recently, York (1987) has shown that the nonconformal modes near a black hole, which give negative actions, seem to suggest the necessity of topological fluctuations. In Section 4, we suggest another possibility.

There is another problem with this program as formulated presently. TET* is not a self-suppporting structure, being built of truncated octahedra (nonrigid). This means that the dynamic variables $s_{at}$, $p_{at}$, $N_{\alpha t}$, and $N^a_{\alpha t}$ are not sufficient to determine fully a null-strut configuration. There is still some debate over whether structural rigidity is necessary to give a well-defined configuration, but for the time being we take the attidue that rigid null-strut sandwiches are necessary. Spinors might provide the extra dynamic variables necessary for rigidification. In the next section, we discuss how we may write a spinor version of the formalism of this section in a simple and natural way. We also discuss how a lattice version of supergravity might be formulated, where the spinors are new dynamic variables, using the constraints given by (11) and (15).

## 3. SPINORS IN NULL-STRUT GEOMETRODYNAMICS

The usefulness of spinors in general relativity is widely accepted, but the methods for including them are varied. General relativity may be rewritten in terms of spinors, as shown by the covariant methods of Penrose, or, alternatively, by the dual spinor canonical approach of Ashtekar (1986a,b). Spinors may also be added to the formalism of general relativity, as is done in supergravity (Freedman and van Nieuwenhuizen, 1976). Certainly, a spinor formulation of null-strut calculus might provide new insight into the structure and implementation of null-strut calculus and give new geometric interpretations of the continuum spinor methods mentioned. We develop a lattice spinorial structure based on null-strut calculus below. The usefulness in rigidification will be outlined and a comparison to the continuum theory of Ashtekar will be given. We also outline how a lattice version of supergravity might be developed using the lattice constraints developed in the last section.

Null-strut lattices allow the inclusion of spinors in a simple and natural way. Consider the comparison, given in Figure 3, of a light-cone and the null-strut equivalent: the fluted cone. This figure shows how the triangles formed by two null-struts emanating from a given vertex in TET, along with the link in TET* between them, may be replaced by null Lorentz spinors in the Penrose interpretation. One of the two null-struts forms the "flagpole" and the link in TET* is a vector in the "flag plane." The resulting structure has new degrees of freedom, namely the phase angles, which, given a set of dynamic constraints for these spinors, might be used to "rigidify" the null-strut lattice. Let us consider these spinors in more detail.

Consider a vertex in the TET($t$) lattice $X$. There are 24 null-struts from this vertex connecting to the 24 vertices of a truncated octahedron in F-TET*($t$). We label these null-strut four-vectors from $X$ as $y_i^\mu(X)$, where $i=1,\ldots,24$; $\mu=1,\ldots,4$. Six spinors may be associated with the six triangles in the entourage of one of these null-struts $y_{i'}^\mu(X)$ (see Figure 4), each one having this null-strut as its "flagpole" and having a spacelike vector, from either TET; $x_{i'j}^\sigma(X)$ or TET*; $k_{i'k}^\nu(X)$ ($j=1,\ldots,3$; $k=1,\ldots,3$; $\sigma=\nu=1,\ldots,4$), defining the plane of the "flag" (see Figure 4).

Let us discuss the spinors that use $k_{i'k}^\nu$ to define the "flag" first, since they are the ones that are crucial for fixing the TET* lattice. We can express the null "flagpole" vector $y_{i'}^\mu(X)$ in terms of a 2-spinor form $Y_{i'}^{A\dot{U}}$ such that

$$y_{i'}^\mu(X) = -\tfrac{1}{2}\sigma^\mu_{A\dot{U}}\, Y_{i'}^{A\dot{U}}(X) \tag{30}$$

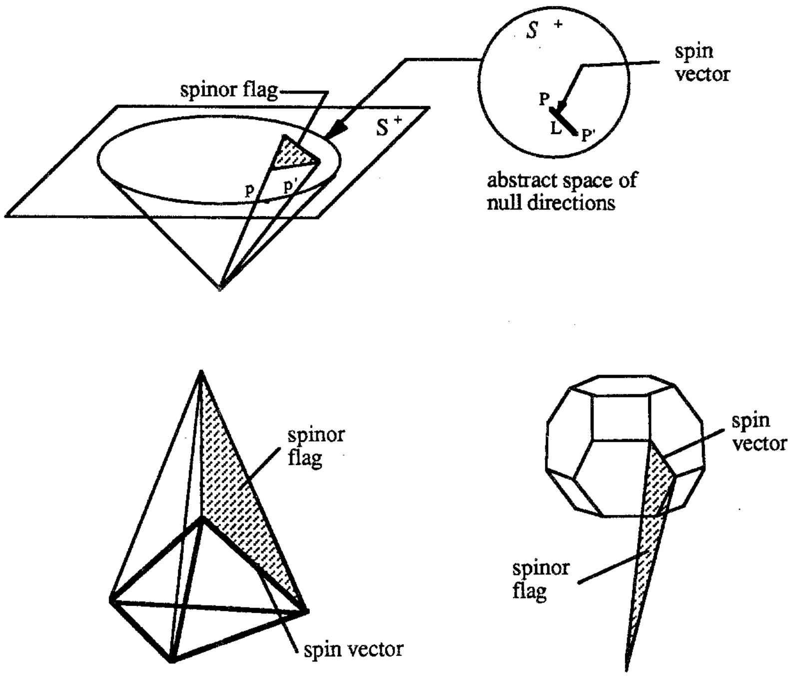


**Fig. 3.** The definitions of a Lorentz spinor and a spin vector on a light cone and the analogous definitions on tne null-strut light cones (wigwam and fluted cone). The triangles formed from two null-struts from the same vertex and a spacelike link give us the "flag plane" of the spinor. One of the null-struts is the "flagpole" of the spinor. The spin vector is a generalization of the spacelike link.

where $\sigma^{\mu}_{A\dot{U}}$ are the Pauli spin matrices, and the dots and capital letters near the end of the alphabet are used to distinguish those components that transform according to the complex conjugate of the Lorentz transformation, as we will discuss below. Since the "flagpole" is null, we may write $Y_{i'}^{A\dot{U}}$ as

$$Y_{i'}^{A\dot{U}}(X) = \xi^{A}_{i'k}(X)\bar{\xi}^{\dot{U}}_{i'k}(X) \qquad \text{(no sum on } i', k) \tag{31}$$

where $\xi^{A}_{i'k}$ is a basis 1-spinor. The bar signifies the complex conjugate. Similarly, we may write the spacelike unit vectors $\hat{k}^{\nu}_{i'k}$ along $k^{\nu}_{i'k}$ in their 2-spinor form $\hat{K}^{B\dot{V}}_{i'k}(X)$ such that

$$\hat{k}^{\nu}_{i'k}(X) = -\tfrac{1}{2}\sigma^{\nu}_{B\dot{V}}\hat{K}^{B\dot{V}}_{i'k}(X) \tag{32}$$

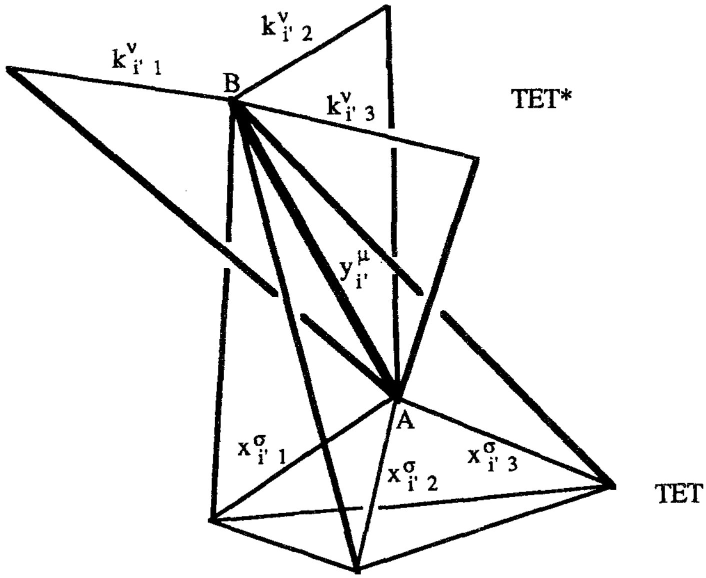


**Fig. 4.** The triangles in the entourage of the null-strut $AB$, interpreted as spinor flags. Each triangle is composed of two null-struts (bold lines), and a spacelike link in either the configuration lattice TET ($x^{\sigma}_{i'l}$), or the momentum lattice TET* ($k^{\nu}_{i'k}$). The null-strut lattice may be interpreted as a network of 2-spinors (and a dual network), thus providing a method for developing a spinor formulation of null-strut calculus. The spacelike links may be generalized to spin vectors. This could provide a way to generalize the spin network theory of three-dimensional quantum spacetime to four dimensions.

Since $\hat{k}^{\nu}_{i'k}$ is spacelike, we may write $\hat{K}^{B\dot{V}}_{i'k}(X)$ as

$$\hat{K}^{B\dot{V}}_{i'k}(X) = \xi^{B}_{i'k}(X)\bar{\eta}^{\dot{V}}_{i'k}(X) + \eta^{B}_{i'k}(X)\bar{\xi}^{\dot{V}}_{i'k}(X) \qquad \text{(no sum on } i'k) \tag{33}$$

where $\eta^{V}_{i'k}$ is the other basis 1-spinor. We shall not refer to the basis 1-spinors again, since, given that we are in a nonorthonornal affine tensor basis, the basis spinors cannot both be interpreted naturally on the null strut lattice at the same time.

Since the spacetime is locally Lorentz-invairant, 2-spinors $X'^{A\dot{U}}$ emanating from a vertex in TET are related by Lorentz transformations $L^{A}_{B}$

$$X'^{A\dot{U}} = L^{A}_{B} X^{B\dot{V}} \bar{L}^{\dot{U}}_{\dot{V}} \tag{34}$$

where $\bar{L}^{\dot{U}}_{\dot{V}}$ is the complex conjugate of the Lorentz transformation (no transpose). The set of Lorentz transformations on the fluted cone contains the information about the angles of the corresponding truncated octahedron in TET* through the spinor phase angles, thus fixing the rigidity. We speculate that these transformations (or the spinors themselves) might be treated as new dynamic variables, which along with $s_{at}$, $p_{at}$, $N_{\alpha t}$, and $N^{a}_{\alpha t}$ provide the necessary structure for the canonical lattice gravity program outlined in Section 2.

Let us now consider an alternative way to reformulate the null-strut lattice in terms of spinors, such that a single null-strut sandwich may be completely written in terms of spinor variables. Using the basis spinors mentioned above, we can write down the following 2-spinor:

$$K_{ik}^{A\dot{U}}(X) = |k_{ik}|[\xi_{ik}^{A}(X)\bar{\eta}_{ik}^{\dot{U}}(X) + \eta_{ik}^{A}(X)\bar{\xi}_{ik}^{\dot{U}}(X)] \tag{35}$$

where $|k_{ik}|$ is the length of the link in TET*, $k_{ik}^{\nu}$. The $K_{ik}^{A\dot{U}}$ is the spinor formulation of $k_{ik}^{\nu}$, and contains all of the information of $k_{at}$, plus information to fix the angles of the truncated octahedra.

Now, consider the past-pointing light cone built of the four past-pointing null-struts from a vertex in TET* $X'$, connected to the vertices of a tetrahedron in TET (a wigwam). A set of basis spinors may be assigned to the triangles in the entourage of one of these null-struts exactly as above. Let us denote this basis as $(\xi_{jl}^{\prime A}(X'),\ \eta_{jl}^{\prime \beta}(X'))$. Here $j$ is an index which specifies which null-strut from $X'$ is the "flagpole" and $l$ specifies which link in TET defines the "flag plane." Therefore, we may define a 2-spinor, similar to that in (28), as

$$X_{jl}^{A\dot{U}}(X') = |x_{jl}|[\xi_{jl}^{\prime A}(X')\bar{\eta}_{jl}^{\prime \dot{U}}(X') + \eta_{jl}^{\prime A}(X')\bar{\xi}_{jl}^{\prime \dot{U}}(X')] \tag{36}$$

where $|x_{jl}|$ is the length of the vector $x_{jl}^{\mu}$ in TET describing a link. The $X_{jl}^{A\dot{U}}(X')$ contains all information on the structure of TET. The phase angles are trivially related, through trigonometric identities, since the tetrahedra of TET are self-supporting. Again, we will not consider the basis 1-spinors, for the same reasons given above. We may now, given a set of dynamic constraints, use $K_{ik}^{A\dot{U}}(X)$, $X_{jl}^{A\dot{U}}(X')$, the lapse function $N_{\alpha t}$, and the shift vector $N^{a}_{\alpha t}$ as the dynamic variables in our canonical lattice quantum gravity formulation. This structure has the advantage of conceptual simplicity, being a spinor network.

Now, consider the relationship between this structure and the continuum spinor formulation of Ashtekar. This analogy gives new insight into the geometric content of the Ashtekar formalism and could provide new methods of numerical solution of both classical and quantum gravity. The first spinor dynamic variable in the Ashtekar formalism is a densitized soldering form $\tilde{\tau}^{a}_{AB}$ from the space of trace-free 2-spinors to the space of vector densities of weight one on the spacelike hypersurface. It is related to the metric $q_{ab}$ on the spacelike hypersurface by the relation

$$(\det q)q_{ab} = -\tilde{\tau}_{A}^{aB}\,\tilde{\tau}_{B}^{bA} \tag{37}$$

Clearly, the variable $X_{jl}^{A\dot{U}}$ is close to a lattice analog to $\tilde{\tau}^a_{AB}$ since

$$|x_{jl}|^2 = -\tfrac{1}{2}X_{jlA\dot{U}}X_{jl}^{A\dot{U}} \tag{38}$$

where $|x_{jl}|^2$, the squared length of the link $l$ in TET, is the lattice analog of the metric. Densitizing $X_{jl}^{A\dot{U}}$ requires dividing by an average volume element $V_\alpha$ of the tetrahedrons containing link $l$. Call the densitized variable $\tilde{X}_{jl}^{A\dot{U}}$. Then (38) becomes

$$V_\alpha |x_{jl}|^2 = -\tfrac{1}{2}\tilde{X}_{jlA\dot{U}}\tilde{X}_{jl}^{A\dot{U}} \tag{39}$$

in direct analogy with (37).

The second dynamic variable in the Ashtekar formalism is a connection on the hypersuface $D_a$ over the space of 1-spinors, given by

$$D_a\lambda_B = \partial_a\lambda_B + A^B_{aA}\lambda_B \tag{40}$$

where $\partial_a$ is a fixed flat connection on the hypersurface, and $A_a^{AB}$ is given by

$$A_a^{AB} = \Gamma_a^{AB} + \frac{i}{\sqrt{2}}\pi_a^{AB} \tag{41}$$

where $\Gamma_a^{AB}$ is the unique connection which annihilates $\tau_a^{AB}$, with $\tau_a^{AB}$ the nondensitized soldering form related to $\tilde{\tau}^a_{AB}$ by $\tilde{\tau}^a_{AB} = (\det q)^{-1/2}\tau^a_{AB}$, between the space of trace-free 2-spinors and the space of tangent vectors of the spacelike hypersurface. $\tau_a^{AB}$ satisfies the relation

$$q_{ab} = -\tau_A^{aB}\tau_B^{bA} \tag{42}$$

$\pi_a^{AB}$ is related to the extrinsic curvature $k_{ab}$ by the relation

$$k_{ab} = \pi^{\ B}_{(aA)}\tau^{\ A}_{b)B} \tag{43}$$

We may use $A_a^{AB}$ as the dynamic variable instead of $D_a$.

The lattice variable $K_{ik}^{A\dot{U}}$, since it is related to the TET* link length $L$, contains information about the extrinsic curvature. The direct relationship between $K_{ik}^{A\dot{U}}$ and $A_a^{AB}$ is certainly quite complicated, given the complicated relationship between $L$ and the extrinsic curvature on the Marionette. The Gauss law constraint of the Ashtekar formalism

$$D_a\tilde{\tau}_A^{aB} = 0 \tag{44}$$

has a very simple lattice interpretation and can be used to interpret the analog of $A_a^{AB}$ on the lattice, $A_a^{A\dot{U}}$. Since we now know how to parallel-transport on the lattice (Schild's ladder), and we know that $\tilde{\tau}^a_{AB}$ has a direct relationship with a link in TET, $A_a^{A\dot{U}}$ may be interpreted as follows:

Convert a 2-spinor $X_{jl}^{A\dot{U}}$ of a TET link from the wigwam at $P$ of a Marionette to the spacelike vector $x^\mu_{ij}$ and parallel-transport this vector to the other wigwam at $P+dP$. Convert this vector back to its spinor form.

We have now performed parallel transport on the spinor $X_{jl}^{A\dot{U}}$. Now, $A_a^{A\dot{U}}$ is related to the Lorentz transformation necessary to rotate this parallel-transported spinor in to a spinor of the wigwam at $P+dP$. There are many unanswered questions in this investigation and work is continuing toward developing a complete formalism. It should also be noted that Finkelstein and Rodriguez (1986) have been attempting to develop a quantum manifold with a free Clifford algebra. The spinor network structure presented above is such a Clifford algebra and provides a compelling avenue toward achieving their program.

A compelling structure for defining a general relativistic theory with spinors is provided through the "square root of general relativity" formalism developed by Tabensky and Teitelboim (1977). This formalism ties the constraints of supergravity to the natural introduction of spinors into general relativity, thus yielding a theory that is more than just a reformulation of ordinary null-strut calculus, as are the formulations given above. The formalism was motivated by Dirac (1958), who showed that when the constraints for a Klein–Gordon particle (which is spinless) are reduced by a "square root" procedure, from quadratic in the momenta to linear in the momenta, the result is the Dirac equation for the electron. As a result of the "square root" procedure, new dynamic variables obeying anticommutation relations were introduced (Dirac spinors), which were associated with the spin of the electron. Einstein's theory is also quadratic in the momenta. Teitelboim performed a similar "square root" procedure on the Hamiltonian and momentum constraints of general relativity. The procedure is outlined below.

The first step is to reformulate the canonical form of general relativity in terms of orthonormal frames (tetrads), which are required if a coupling to spinor fields is to be accomplished. We introduce at each point in space a triad of vectors $\lambda_i^a$ $(a, i=1,3)$, which satisfy the completeness and orthogonality relations

$$\lambda_{ai}\lambda_{aj} = g_{ij} \tag{45}$$

and

$$\lambda_{ai}\lambda^i_{\ b} = \delta_{ab} \tag{46}$$

where $g_{ij}$ is the spatial metric (with conjugate momenta $\pi^{ij}$). We also introduce the conjugates of $\lambda_{ai}$, called $\pi^i_{\ a}$, which satisfy the commutation relations

$$|\lambda_{ai}(x), \pi^j_b(x')| = i\delta_{ab}\delta^j_i\delta(x, x') \tag{47}$$

The tetrad is formed from the triad $\lambda_{ai}$ and the normal to the spacelike hypersurface. This allows us to make arbitrary localized spatial rotations on the hypersurface without surface deformations.

The constraints of this theory are composed of the generators of normal and tangential surface deformations, which are

$$H_0 = G_{ijkl}\pi^{ij}\pi^{kl} - g^{1/2}R \approx 0 \tag{48}$$

and

$$H_1 = -2\pi^j_{i|j} \approx 0 \tag{49}$$

respectively, and the generator of local rotations $\gamma_{ab}$, which is

$$\gamma_{ab} = \pi^i_a\lambda_{bi} - \pi^i_b\lambda_{ai} \approx 0 \tag{50}$$

$G_{ijkl}$ is the "supermetric" given by

$$G_{ijkl} = \tfrac{1}{2}g^{-1/2}(g_{ik}g_{jl} + g_{il}g_{jk} - g_{ij}g_{kl}) \tag{51}$$

$R$ is the curvature of $g_{ij}$, and the vertical slash in (49) denotes spatial covariant differentiation. $\pi^{ij}$ is related to $\pi^i_a$ by

$$\pi^{ij} = \tfrac{1}{2}(\pi^i_a\lambda^j_a + \pi^j_a\lambda^i_a) \tag{52}$$

To perform the "square root" procedure on the constraints given by (48)-(50), we must form a function which is linear in the momenta

$$\Delta_A(x) = \Gamma_{Aij}\pi^{ij} + V_A \tag{53}$$

$\Gamma_{Aij}$ and $V_A$ are independent of $\pi^i_a$ and must be chosen so that the anticommutator $\{\Delta_A(x), \Delta_B(x')\}$ is a linear functional of the constraints (48)-(50), with the possible modification by terms containing $\Gamma_{Aij}$, which disappear in the continuum limit. The simplest candidates for $\Gamma_{Aij}$ and $V_A$ turn out to be

$$\Gamma_{Aij} = \tfrac{1}{2}(\gamma_i\psi_j + \gamma_j\psi_i) \tag{54}$$

and

$$V_A(x) = g^{1/2}[\gamma^i, \gamma^j]\nabla_i\psi_j + \text{trilinear terms in the spinors} \tag{55}$$

where $\psi_j$ is a Majorana vector-spinor, and $\nabla_i\psi_j$ is the torsion-free covariant derivative of $\psi_j$. The anticommutator of $\Delta$ with itself becomes

$$\{\Delta_A(x), \Delta_B(x')\} = \delta(x, x')\gamma^\mu_{AB}H^{\text{NEW}}_\mu \tag{56}$$

where $H^{\text{NEW}}_\mu$ differs from (48) and (49) by spinor terms. Finally, the new constraints are given by the supersymmetry generator

$$\Delta_A(x) = 0 \tag{57}$$

the surface deformation generator

$$H^{\text{NEW}}_\mu = 0 \tag{58}$$

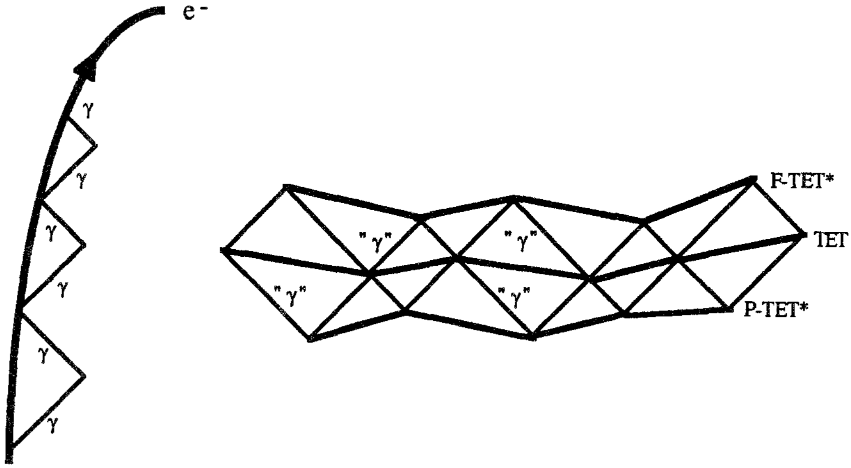


**Fig. 5.** The result of measuring any component of the velocity for a free Dirac electron yields the speed of light. The analogy between this *Zitterbewegung* of the Dirac electron and the lightlike propagation of the three-geometry in null-strut calculus is displayed in this diagram. The development of a Clifford algebra ("γ") for null-strut calculus may be obtained, in analogy with the Clifford algebra of the Dirac electron (γ), by finding the "square-root" constraints of the theory.

and the rotation generator

$$\gamma_{ab}^{\mathrm{NEW}} = 0 \tag{59}$$

where $\gamma_{ab}^{\mathrm{NEW}}$ differs from (50) by rotations of the spinor variables. The algebra for the constraints of this theory is the same as supergravity. A similar procedure might be possible on the null-strut lattice version of the constraints, given by (11) and (15), to obtain the lattice supergravity constraints. The development of this structure is under investigation.

A further motivation for the "square root" approach is given by the analysis of the motion of a free Dirac electron. Due to the principle of uncertainty, a measurement of any component of the velocity of a free electron leads to the result $\pm c$. In a loose sense, the Dirac electron moves along a path of connected null-struts (see Figure 5), much as the three-geometries in null-strut geometrodynamics propagate in time along a series of null-struts.

## 4. FURTHER IMPLICATIONS OF THE SPIN STRUCTURE OF SPACETIME

In this section, let us indulge in some unbridled speculation on further possible implications of the spinor structure of spacetime discussed in

Section 3. Let us assume, for the sake of argument, that the simplicial structure is fundamental and the continuum is an approximation. Several physicists have given compelling arguments in this direction, including Lee (1984) and Penrose (1970, 1972). This assumption is central to the following presentation.

A most compelling formulation of three-dimensional quantum gravity in terms of spin networks was given by Ponzano and Regge and elaborated on by Hasslacher and Perry (1981) and Lewis (1983). A description of the formalism begins, just as in Euclidean simplicial quantum gravity, with the triangulation of the two-dimensional boundary ($S^2$ at infinity for vacuum or $S^1 \times S^1$ for a canonical ensemble at temperature $\beta^{-1}$), except that the link lengths are restricted to the values $(j_i + 1/2)\hbar$, where $j_i = 0, 1/2, 1, 3/2, 2, \ldots$. This triangulation acts as a spin network. To this triangulation we associate a quantity $Z$ which is the $3nj$-coefficient of the resulting spin network.

Yutsis *et al.* (1960) give a good presentation of the $3nj$-coefficient and we will not attempt to reproduce the whole account here. The $3nj$-coefficient, denoted by

$$\begin{Bmatrix} j_1 & j_2 \cdots j_n \\ l_1 & l_2 \cdots l_n \\ k_1 & k_2 \cdots k_n \end{Bmatrix} \tag{60}$$

(where the $j_i$, $l_i$, and $k_i$ are total angular momentum quantum numbers), is a product of generalized Wigner coefficients, denoted by

$$\begin{pmatrix} j_1 \cdots j_n \\ m_1 \cdots m_n \end{pmatrix}^{A}_{a_1 \cdots a_{n-3}} \tag{61}$$

(where $A$ describes the coupling scheme, and the $a_i$ are the intermediate angular momenta), which are summed over all the magnetic quantum numbers $m_i$. The generalized Wigner coefficients are a symmetrical version of the usual generalised Clebsch-Gordan coefficients for the addition of angular momenta,

$$\begin{aligned} (j_1 m_1 \cdots j_n m_n &| (j_1 \cdots j_n)^A a_1 \cdots a_{n-2} JM) \\ &= (-1)^{f_A(j_1 \cdots j_n^J)} [(a_1) \cdots (a_{n-2})]^{1/2} \\ &\quad \times (-1)^{J-M} (J)^{1/2} \begin{pmatrix} j_1 \cdots j_n & J \\ m_1 \cdots m_n & -M \end{pmatrix}^{A}_{a_1 \cdots a_{n-2}} \end{aligned} \tag{62}$$

where $f_A$ is a function of the addition scheme. The simplest example of a $3nj$-coefficient is the $6j$-coefficient, which is a product of four Wigner coefficients, given by

$$\begin{Bmatrix} j_1 \, j_2 \, j_3 \\ l_1 \, l_2 \, l_3 \end{Bmatrix} = \sum_{m_i n_i} (-1)^{j_1 - m_1 + j_2 - m_2 + j_3 - m_3 + l_1 - n_1 + l_2 - n_2 + l_3 - n_3}$$

$$\times \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & -m_3 \end{pmatrix} \begin{pmatrix} j_1 & l_2 & l_3 \\ -m_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & j_3 \\ n_1 & -n_2 & m_3 \end{pmatrix} \begin{pmatrix} l_1 & j_2 & l_3 \\ -n_1 & -m_2 & -n_3 \end{pmatrix} \quad (63)$$

Figure 6 shows the graphical representation of a $6j$-coefficient. Any $3nj$-coefficient can be written as a product of $6j$-coefficients, using the relation

$$\begin{Bmatrix} j_1 & j_2 \cdots j_n \\ l_1 & l_2 \cdots l_n \\ k_1 & k_2 \cdots k_n \end{Bmatrix} = \sum_r (r)(-1)^{R_n + (n-1)r}$$

$$\times \begin{Bmatrix} j_1 & k_1 & r \\ k_2 & j_2 & l_1 \end{Bmatrix} \begin{Bmatrix} j_2 & k_2 & r \\ k_3 & j_3 & l_2 \end{Bmatrix} \cdots \begin{Bmatrix} j_{n-1} & k_{n-1} & r \\ k_n & j_n & l_{n-1} \end{Bmatrix} \begin{Bmatrix} j_n & k_n & r \\ j_1 & k_1 & l_n \end{Bmatrix} \quad (64)$$

where

$$R_n = \sum_{i=1} (j_i + k_i + l_i) \quad (65)$$

We can use (64) to devise an operation which fills the interior of the boundary with a net of tetrahedra ($6j$-coefficients), each obeying the same rules as the boundary. The resulting simplicial complex is a three-dimensional space. It can be shown that the three-dimensional path integral of quantum gravity is a special case of the $3nj$-coefficient given by $Z$.

Let us consider a simple example where the triangulated two-dimensional boundary is a tetrahedron ($S^2$ topology). Therefore $Z$ is a $6j$ symbol. The link length from vertex $j$ to vertex $k$ will be denoted by $a_{jk}$.

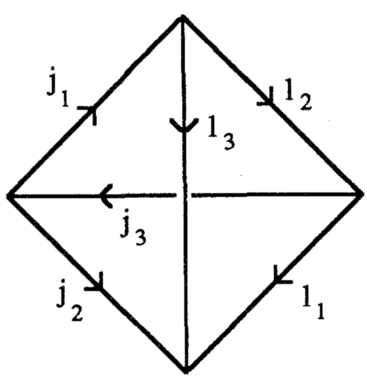


**Fig. 6.** Graphical representation of the $6j$ symbol. This is the basic building block of three-dimensional quantum spacetime in the spin network theory.

The asymptotic form of the 6$j$ symbol for large values of $j_i$, with the usual simplicial inequalities satisfied, is given by

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \end{Bmatrix} \approx \left(\frac{\hbar^3}{12\pi V}\right)^{1/2} \cos\left[\left(\sum_{k,k} \frac{1}{\hbar} a_{jk}\theta_{jk}\right) + \frac{\pi}{4}\right] \tag{66}$$

where $V$ is the volume of the tetrahedron, and $\theta_{jk}$ is the angle between the outward unit normals to the faces that are separated by link $jk$. If we analytically continue this expression to the regime where the simplicial inequalities are violated (the tetrahedron cannot be produced in flat $R^3$), we get the expression

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ J_4 & J_5 & J_6 \end{Bmatrix} \approx \pm\left(\frac{\hbar^3}{48\pi V}\right)^{1/2} \exp\left(-\frac{1}{\hbar}\left|\sum_{j,k} a_{jk} \operatorname{Im} \theta_{jk}\right|\right) \tag{67}$$

where $\theta_{jk}$ and $V$ are the analytic continuations of the corresponding expressions when the simplicial inequalities are satisfied. This regime corresponds to quantum coupling of angular momentum and a nontrivial topology for the interior of the tetrahedron.

The 6$j$ symbols obey an identity introduced by Biedenharn (1953) and Elliott (1953) given as

$$\begin{Bmatrix} J_1 & J_2 & J_3 \\ l_1 & l_2 & l_3 \end{Bmatrix}\begin{Bmatrix} J_1 & J_2 & J_3 \\ l_4 & l_5 & l_6 \end{Bmatrix} = \sum_x (-1)^\phi (2x+1) \times \begin{Bmatrix} J_1 & l_5 & l_6 \\ x & l_3 & l_2 \end{Bmatrix}\begin{Bmatrix} l_4 & J_2 & l_6 \\ l_3 & x & l_1 \end{Bmatrix}\begin{Bmatrix} l_4 & l_5 & J_3 \\ l_2 & l_1 & x \end{Bmatrix} \tag{68}$$

where

$$\phi = x + \sum l_i + \sum j_i \tag{69}$$

and $x$ runs over an allowed domain. We must note that this identity applies even if the 6$j$ symbols on the left-hand side are in the classical domain, and the 6$j$ symbols on the right-hand side are in the quantum domain. There is also an identity which replaces a 6$j$ symbol by four 6$j$ symbols,

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \sum_{\substack{\text{allowed domain} \\ \text{of } w,x,y,z}} (2w+1)(2x+1)(2y+1)(2z+1) \times \begin{Bmatrix} a & b & c \\ x & y & z \end{Bmatrix}\begin{Bmatrix} f & e & a \\ y & z & w \end{Bmatrix}\begin{Bmatrix} d & b & f \\ z & w & x \end{Bmatrix}\begin{Bmatrix} c & e & d \\ w & x & y \end{Bmatrix} \tag{70}$$

The geometric interpretation of this identity is given in Figure 7. The general procedure can now be presented as follows.

If we have a $3nj$-coefficient referring to a two-dimensional boundary, we can decompose it into a sum of products of 6$j$ symbols. These 6$j$ symbols correspond to the $n/2$ tetrahedra (simplicial inequalities are not necessarily

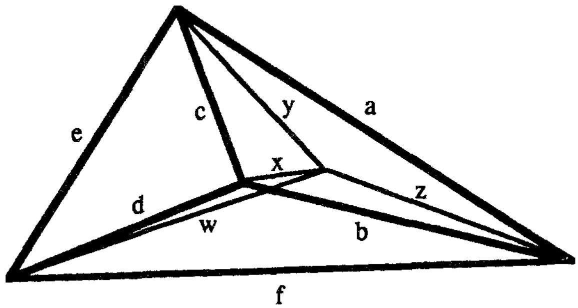


**Fig. 7.** Graphical representation of the quantum cobordism identity. Every 6*j* symbol can be decomposed into the sum of products of four 6*j* symbols. It allows an approach to the continuum by repeated application of this decomposition procedure. The resulting theory contains a scale invariance (with renormalization parameter) not present in ordinary simplicial theories.

satisfied) in the tesselation of the interior of the boundary. We can apply (70) to each of the $n/2$ tetrahedra. Applying (70) a total of $p$ times to the tetrahedra yields $m = 2^{2^{p-1}}$ tetrahedra and a successively finer mesh, but does not effect the value of $Z$. This is known as a quantum cobordism operation. As $p$ approaches infinity, the mesh approaches the continuum three-dimensional space.

We can examine the semiclassical limit of the $3nj$-coefficient by letting $\hbar$ go to zero, letting $p$ go to $\infty$, and keeping the edge lengths on the boundary fixed. As we let $p$ go to $\infty$, the mesh becomes fine enough to tesselate any three manifold with the simplicial inequalities satisfied. Letting $\hbar$ approach zero while keeping the edge lengths fixed is equivalent to letting $j_i$ go to $\infty$. We can therefore replace all of the $6j$-coefficients in the decomposition by their asymptotic form. After splitting the cosines of (66) into exponentials, we obtain

$$Z \sim \int \prod_i dx_i \, (2x_i + 1) \exp\left( \frac{i}{\hbar} \sum_{\text{tetrahedra}, j, k} a_{jk} \theta_{jk} + 2^{m-2} \pi \right) \tag{71}$$

where $x_i$ are variables assigned to the internal links. This is just the path integral for three-dimensional simplicial quantum gravity.

We see from the development above that the $3nj$-coefficient is a much more general structure than the path integral. This richer structure has the following desirable properties:

1. The formalism gives a physical justification for the inclusion of a renormalization cutoff: $\frac{1}{2}\hbar$.
2. The discrete theory is a consistent generalization of the continuum theory.

3. The theory contains a natural renormalization parameter $p$, which comes about from a fractal approach to the continuum.
4. Spacetime foam is automatically included in the theory by the relaxation of the simplicial inequalities.
5. Calculations can be done without the evaluation of path integrals, since the general expression for the $3nj$ coefficient is strictly combinatorial in nature.
6. The theory of quantum gravity is reduced to very simple building blocks with very simple laws: spins and their addition.
7. Consider the evaluation of the semiclassical path integral given by (71) using the Monte Carlo method. The restriction of the link lengths to nonzero integer multiples of $\frac{1}{2}\hbar$ means that if the measure has a sufficiently strong dependence on the link lengths in the denominator (at least exponential), configurations with large negative curvatures would be suppressed. This is because these configurations would require large link length values, unlike the continuum link length case, where large curvatures can occur due to link lengths becoming small or large. We therefore have a possible solution to the positive-definiteness problem.

Unfortunately, the generalization of the theory to four dimensions is unclear and very little work has been done to develop the theory further. The spinor network formulation discussed above may provide the natural

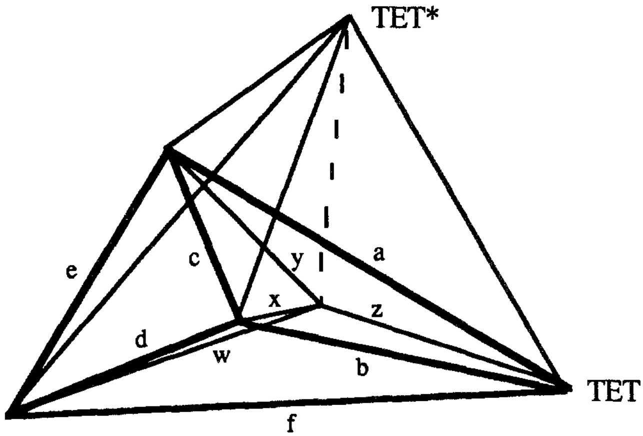


**Fig. 8.** A possible graphical representation of a quantum cobordism identity in four dimensions. The four-simplex (wigwam) is decomposed into the sum of products of four wigwams. A similar identity might be formulated for the fluted cones. This suggests that four-dimensional quantum spacetime may be formulated as two spin networks, dual to each other.

four-dimensional analog to the spin network formulation, since the spinor is the natural extension of the spin vector. If we consider the Penrose interpretation of the Lorentz spinor, as used in our null-strut formalism (see Figure 3), the links in TET* and TET can be reinterpreted as spin vectors. Let us restrict these links to the discrete values $(j_i+1/2)\hbar$. This means that the three-dimensional hypersurfaces of TET($t$) and TET*($t$) are interpreted as spin networks in the sense given previously, except that there are constraints between the spin network of TET($t$) and the spin network of TET*($t$). The naive assignment of $3nj$-coefficients to the spacelike hypersurfaces is probably not reasonable.

The constraints between the spin networks might be thought of as being due to the interaction of integer-spin objects (null-struts) with the half-integer-spin objects of the spin networks. Finkelstein and Rodriguez have mentioned the fact that the simplices of a quantum manifold should have both integer and half-integer spins it its spectrum. Using rules for the interaction of inter-spin objects and half-integer-spin objects, it might be possible to form an analog of the $3nj$-coefficient which would reduce to four-dimensional simplicial quantum gravity (or some reasonable generalization) in the semiclassical limit. A possible generalization of the quantum cobordism, which fills the fluted cones (and the wigwams), is exhibited in Figure 8. Certainly, the recovery of part of the beautiful properties listed above makes the effort expended in this program worthwhile. Meanwhile, a two-dimensional simplicial quantum gravity program is being implemented to evaluate the semiclassical path integral with discrete link lengths and several avenues for generalizing spin networks are being studied.

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